Riemannian Interior Point Methods for Constrained Optimization on Manifolds

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Overview

Riemannian Manifold

Riemannian manifold M is a locally linearizable set, equipped with a smoothly-varying inner product $\langle \cdot, \cdot \rangle_x$ on the tangent spaces T_xM.

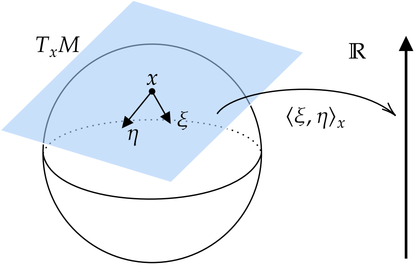

Figure: Manifold of unit sphere, $M = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$.

Riemannian Optimization

Given $f : M \to \mathbb{R}$, solve

$$\min_{x \in M} f(x)$$

where M is a Riemannian manifold.

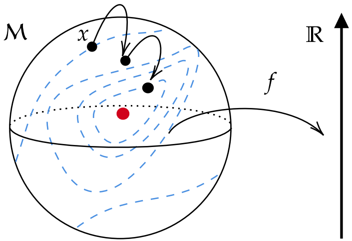

Figure: Iteration on unit sphere

Unconstrained problem on manifold.

1. Stiefel manifold, $\mathrm{St}(n,k) = \{X \in \mathbb{R}^{n\times k} : X^\top X = I\}$.
2. Fixed rank manifold, $\mathbb{R}_r^{m\times n} = \{X \in \mathbb{R}^{m\times n} : \mathrm{rank}(X) = r\}$.

Riemannian version of classical methods (2002-)

steepest decent, conjugate gradient, trust region, BFGS, proximal gradient, ADMM and more.

Applications

Stiefel manifold, $\mathrm{St}(n,k) = \{X \in \mathbb{R}^{n\times k} : X^\top X = I\}$.

PCA:

$$\min_{X\in\mathrm{St}(n,k)} -\mathrm{trace}(X^\top A^\top A X).$$

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Low-rank matrix completion:

$$\min_{X\in \mathbb{R}_r^{m\times n}} \sum_{(i,j)\in\Omega} (X_{ij} - M_{ij})^2.$$

More Requirements

Stiefel manifold, $\mathrm{St}(n,k) = \{X \in \mathbb{R}^{n\times k} : X^\top X = I\}$.

Nonnegative PCA:

$$\min_{X\in \mathrm{St}(n,k)} -\operatorname{trace}(X^\top A^\top A X)$$

$$\text{s.t. } X \geq 0$$

Fixed rank manifold, $\mathbb{R}_r^{m\times n} = \{X \in \mathbb{R}^{m\times n} : \operatorname{rank}(X) = r\}$.

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$$\text{s.t. } X \geq 0$$

New Topic — Riemannian Constrained Optimization Problem

We consider

$$\min_{x \in \mathbb{M}} \quad f(x) \qquad \text{s.t.} \quad h(x) = 0, \text{ and } g(x) \leq 0, \tag{RCOP}$$

where $\mathbb{M}$ is a Riemannian manifold, $f : \mathbb{M} \to \mathbb{R}$, $h : \mathbb{M} \to \mathbb{R}^l$, and $g : \mathbb{M} \to \mathbb{R}^m$.

Riemannian version of optimality conditions:

- KKT conditions; Second-order conditions [Yang et al., 2014];
- More constraint qualifications (CQ) [Bergmann and Herzog, 2019];
- Sequential optimality conditions [Yamakawa and Sato, 2022].

Riemannian version of classical algorithms:

- Augmented Lagrangian Method [Liu and Boumal, 2020, Yamakawa and Sato, 2022];
- Exact Penalty Method [Liu and Boumal, 2020];
- Sequential Quadratic Programming Method [Schiela and Ortiz, 2020, Obara et al., 2022].
- In this talk, we consider Riemannian version of Interior Point Method.

Riemannian Interior Point Methods (RIPM)

Retraction — moving on a manifold

A **retraction** R maps tangent vectors back to the manifold.
$R_x : T_xM \to M$ for any x.

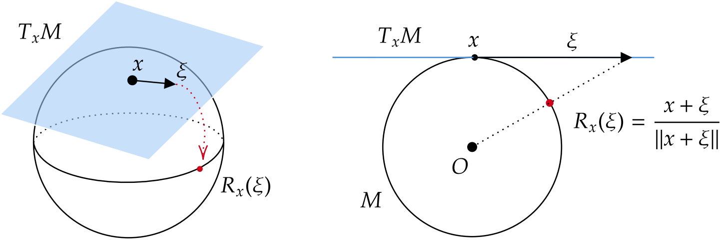

Figure: A retraction on $M = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$.

Iteration on manifolds:

Euclidean	Riemannian
$x_{k+1} = x_k + \alpha_k \xi_k$	$x_{k+1} = R_{x_k}(\alpha_k \xi_k)$

Riemannian gradient — a vector field

Riemannian gradient, $\operatorname{grad} f(x)$, is the direction (tangent vector) of steepest ascent in tangent space at x:

$$\frac{\operatorname{grad} f(x)}{\|\operatorname{grad} f(x)\|} = \underset{\xi \in T_x M : \|\xi\| = 1}{\arg\max} \left(\lim_{\alpha \to 0} \frac{f(R_x(\alpha\xi)) - f(x)}{\alpha} \right).$$

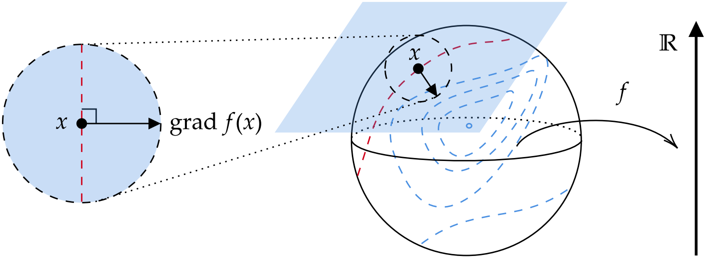

Figure: $\operatorname{grad} f(x)$ is perpendicular to the contour line of f on $M = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$.

Note that $x \mapsto \operatorname{grad} f(x)$ is a **vector field** on M.

Covariant derivative & Hessian & Newton method

Covariant derivative of a vector field F:

Riemannian connection ↓

$$\nabla F(x) : T_xM \to T_xM, \ \textit{linear operator}.$$

↑ general vector field

Specially, $\operatorname{Hess} f(x) \triangleq \nabla \operatorname{grad} f(x)$ is called **Riemannian Hessian**.

Covariant derivative & Hessian & Newton method

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general vector field

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Riemannian Newton method: To find singularity $x^* \in M$ such that $F(x^*) = 0_{x^*}$.
Solve a linear system on $T_{x_k}M \ni v_k$:

$$\nabla F(x_k) v_k = -F(x_k), \tag{1}$$

then $x_{k+1} = R_{x_k}(v_k)$.

Riemannian Interior Point Methods (RIPM)

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Formulation of RIPM

We consider

$$\min_{x\in\mathbb{M}} \quad f(x) \qquad \text{s.t.} \quad h(x)=0, \text{ and } g(x)\leq 0, \tag{RCOP}$$

where $f:\mathbb{M}\to\mathbb{R}$, $h:\mathbb{M}\to\mathbb{R}^l$, and $g:\mathbb{M}\to\mathbb{R}^m$.

Lagrangian function is

$$\mathcal{L}(x,y,z) \triangleq f(x) + y^T h(x) + z^T g(x). \tag{2}$$

$x \mapsto \mathcal{L}(x,y,z)$ is a real-valued function on $\mathbb{M}$,

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$$\text{s.t.} \quad h(x)=0, \ \text{and } g(x)\leq 0, \qquad \text{(RCOP)}$$

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$x \mapsto \mathcal{L}(x,y,z)$ is a real-valued function on $\mathbb{M}$, then we have

- $\operatorname{grad}_x \mathcal{L}(x,y,z) =$
  $\operatorname{grad} f(x) + \sum_{i=1}^{l} y_i \operatorname{grad} h_i(x) + \sum_{i=1}^{m} z_i \operatorname{grad} g_i(x)$,
- $\operatorname{Hess}_x \mathcal{L}(x,y,z) =$
  $\operatorname{Hess} f(x) + \sum_{i=1}^{l} y_i \operatorname{Hess} h_i(x) + \sum_{i=1}^{m} z_i \operatorname{Hess} g_i(x)$.

KKT Vector Field

Riemannian KKT conditions [Liu and Boumal, 2020] are

$$\begin{cases} \operatorname{grad}_x \mathcal{L}(x,y,z) = 0_x, \\ h(x) = 0, \\ g(x) \leq 0, \\ Zg(x) = 0, \ (Z := \operatorname{diag}(z_1,\ldots,z_m)) \\ z \geq 0. \end{cases} \tag{3}$$

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Definition (KKT Vector Field, L.2022)

Using $s := -g(x)$, the above becomes

$$F(w) \triangleq \begin{pmatrix} \operatorname{grad}_x \mathcal{L}(x, y, z) \\ h(x) \\ g(x) + s \\ ZSe \end{pmatrix} = 0_w := \begin{pmatrix} 0_x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and } (z, s) \geq 0, \tag{4}$$

where $w := (x, y, z, s) \in \mathscr{M} \triangleq \mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$. Note that $T_w\mathscr{M} \equiv T_x\mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$.

Covariant Derivative of KKT Vector Field

For each $x \in \mathbb{M}$, we define

$$H_x : \mathbb{R}^l \to T_x\mathbb{M}, \quad H_x v \triangleq \sum_{i=1}^{l} v_i \operatorname{grad} h_i(x). \tag{5}$$

Hence, the adjoint operator is

$$H_x^* : T_x\mathbb{M} \to \mathbb{R}^l, \quad H_x^*\xi = \left[\langle \operatorname{grad} h_1(x), \xi \rangle_x, \cdots, \langle \operatorname{grad} h_l(x), \xi \rangle_x\right]^T. \tag{6}$$

Lemma (L. 2022)

The linear operator $\nabla F(w) : T_w\mathscr{M} \to T_w\mathscr{M}$ is given by

$$\nabla F(w)\Delta w = \begin{pmatrix} \operatorname{Hess}_x \mathcal{L}(w)\Delta x + H_x\Delta y + G_x\Delta z \\ H_x^*\Delta x \\ G_x^*\Delta x + \Delta s \\ Z\Delta s + S\Delta z \end{pmatrix}, \tag{7}$$

where $\Delta w = (\Delta x, \Delta y, \Delta s, \Delta z) \in T_x\mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m \equiv T_w\mathscr{M}$.

Riemannian Interior Point Method (RIPM)

Step 0. Initial w_0 with $(z_0, s_0) > 0$.
Step 1. Solve

$$\nabla F(w_k)\Delta w_k = -F(w_k) + \mu_k \hat{e}, \tag{8}$$

where $\hat{e} \triangleq (0_x, 0, 0, e)$.
Step 2. Compute the step sizes α_k such that $(z_{k+1}, s_{k+1}) > 0$.
Step 3. Update:

$$w_{k+1} = \bar{R}_{w_k}(\alpha_k \Delta w_k). \tag{9}$$

Step 4. Shrink $\mu_k \to 0$. Return to 1.

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Theorem (Local Convergence, L. 2022)

Under some standard assumptions.

1. *If $\mu_k = o(\|F(w_k)\|)$, $\alpha_k \to 1$, then $\{w_k\}$ locally, superlinearly converges to w^*.*
2. *If $\mu_k = O(\|F(w_k)\|^2)$, $1 - \alpha_k = O(\|F(w_k)\|)$, then $\{w_k\}$ locally, quadratically converges to w^*.*

Global Line Search RIPM Algorithm

- **Merit function:** Choose $\varphi(w) \triangleq \|F(w)\|^2$.
- **Backtracking for step size α_k:**
  1. Centrality conditions.
  2. With a slight abuse of notation, we also let

$$\varphi(\alpha) \triangleq \varphi(\underbrace{\bar{R}_{w_k}(\alpha \Delta w_k)}_{\text{new iterate}}) \text{ for fixed } w_k \text{ and } \Delta w_k, \qquad (10)$$

then $\varphi(0) = \varphi(w_k) =: \varphi_k$ and $\varphi'(0) = \langle \operatorname{grad} \varphi(w_k), \Delta w_k \rangle$. Sufficient decreasing asks

$$\varphi(\alpha_k) - \varphi(0) \leq \alpha_k \beta \varphi'(0).$$

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$$\varphi(\alpha_k) - \varphi(0) \leq \alpha_k \beta \varphi'(0).$$

- **Descent direction:** Let Δw_k be the solution of $\nabla F(w_k) \Delta w_k = -F(w_k) + \rho_k \sigma_k \hat{e}$, then

$$\varphi'(0) < 0 \text{ if we set } \rho_k := s_k^T z_k / m, \sigma_k \in (0, 1).$$

The sequence $\{\varphi_k\}$ is monotonically decreasing.

Global Convergence

Assumptions:

1. the functions $f(x), h(x), g(x)$ are smooth; the set $\{\operatorname{grad} h_i(x)\}_{i=1}^{l}$ is linearly independent in $T_x\mathbb{M}$ for all x; and $w \mapsto \nabla F(w)$ is Lipschitz continuous;
2. the sequences $\{x_k\}$ and $\{z_k\}$ are bounded;
3. the operator $\nabla F(w)$ is nonsingular.

Theorem (Global Convergence, L. 2022)

Let $\{\sigma_k\} \subset (0,1)$ bounded away from zero and one. If Assumptions 1∼3 hold, then $\{F(w_k)\}$ converges to zero; and for any limit point $w^* = (x^*, y^*, z^*, s^*)$ of $\{w_k\}$, x^* is a Riemannian KKT point of problem (RCOP).

Riemannian Interior Point Methods (RIPM)

Dominant cost — Newton equation

Dominant cost is to solve

$$\nabla F(w)\Delta w = -F(w) + \mu \hat{e}, \tag{11}$$

where

$$F(w) = \begin{pmatrix} F_x \triangleq \operatorname{grad}_x \mathcal{L}(x,y,z) \\ F_y \triangleq h(x) \\ F_z \triangleq g(x) + s \\ F_s \triangleq ZSe \end{pmatrix}, \quad \hat{e} \triangleq \begin{pmatrix} 0_x \\ 0 \\ 0 \\ e \end{pmatrix}. \tag{12}$$

Thus, we need to solve the following linear system on $T_x\mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$:

$$\begin{pmatrix} \operatorname{Hess}_x \mathcal{L}(w)\Delta x + H_x \Delta y + G_x \Delta z \\ H_x^* \Delta x \\ G_x^* \Delta x + \Delta s \\ Z\Delta s + S\Delta z \end{pmatrix} = \begin{pmatrix} -F_x \\ -F_y \\ -F_z \\ -F_s + \mu e \end{pmatrix}. \tag{13}$$

Condensed form of Newton equation

It suffices to focus on **condensed form** on $T_x\mathbb{M} \times \mathbb{R}^l$:

$$\mathcal{T}(\Delta x, \Delta y) := \begin{pmatrix} \mathcal{A}_w \Delta x + H_x \Delta y \\ H_x^* \Delta x \end{pmatrix} = \begin{pmatrix} c \\ q \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} \mathcal{A}_w &:= \operatorname{Hess}_x \mathcal{L}(w) + G_x S^{-1} Z G_x^*, \\ c &:= -F_x - G_x S^{-1}\left(Z F_z + \mu e - F_s\right), \quad q := -F_y. \end{aligned} \tag{15}$$

- $\mathcal{A}_w$ is self-adjoint (but may indefinite) on $T_x\mathbb{M}$.
- $\mathcal{T}$ is self-adjoint (but may indefinite) on $T_x\mathbb{M} \times \mathbb{R}^l$. This is a saddle point problems on Hilbert space.
- The Riemannian situation leaves us with no explicit matrix form available.
- A simple approach is to first find the representing matrix $\hat{\mathcal{T}}$ under some basis. (Expensive !)

Krylov subspace methods on Tangent space

An ideal approach is to use iterative methods, such as **Krylov subspace methods** (e.g., Conjugate Gradients method), on $T_x\mathbb{M} \times \mathbb{R}^l$ directly.

For simplicity, we consider the case of only inequality constraints, where Δy vanishes, thus we only needs to

$$\text{solve } \mathcal{A}_w \Delta x = c \text{ for } \Delta x \in T_x\mathbb{M}. \quad (16)$$

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- It only needs to call an abstract linear operator $v \mapsto \mathcal{A}_w v$. (matrix-vector product)
- All the iterates v_k are in $T_x\mathbb{M}$.
- Since operator $\mathcal{A}_w$ is self-adjoint but indefinite, we use Conjugate Residual (CR) method to solve it.

The discussion of above can be naturally extended to the general case.

Riemannian Interior Point Methods (RIPM)

Numerical Experiments

We compare with the other Riemannian methods:[1]

- RALM : Riemannian augmented Lagrangian method.
- REPM(LQH) : Riemannian exact penalty method with smoothing function LQH.
- REPM(LSE) : Riemannian exact penalty method with smoothing function LSE.
- RSQP : Riemannian sequential quadratic programming.
- RIPM (Our method): Riemannian interior point method.

KKT residual is defined by

$$\sqrt{\|\operatorname{grad}_x \mathcal{L}(w)\|^2 + \sum_{i=1}^{m}\{\min(0, z_i)^2 + \max(0, g_i(x))^2 + |z_i g_i(x)|^2\} + \sum_{i=1}^{l} |h_i(x)|^2 + \text{Manvio}(x)},$$

where Manvio measures the violation of manifold constraints.

[1]The numerical experiments were performed in Matlab R2022a on a computer equipped with an Intel Core i7-10700 at 2.90GHz with 16GB of RAM.

Problem I — Nonnegative Low Rank Matrix Approximation (NLRM)

Problem I

[Song and Ng, 2020] proposed

$$\min_{X \in \mathbb{R}_r^{m \times n}} \|A - X\|_F^2 \quad \text{s.t. } X \geq 0,$$

where $\mathbb{R}_r^{m \times n} = \{X \in \mathbb{R}^{m \times n} : \operatorname{rank}(X) = r\}$.

Data setting:

```
B = rand(m, r);
C = rand(r, n);
A = B*C+sigma*randn(m,n);
```

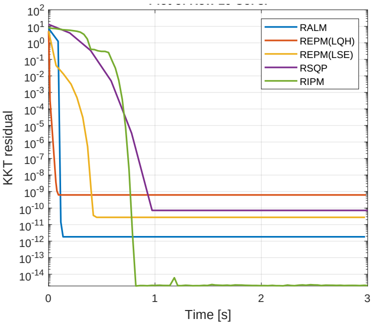

Figure: $m = 10, n = 8, r = 3$ and $\sigma = 0.01$.

Problem II — Projection onto nonnegative Stiefel manifold

Problem II Given $C \in \mathbb{R}^{n\times k}$, we consider

$$\min_{X\in \mathrm{St}(n,k)} \|X - C\|_F^2, \quad \text{s.t. } X \geq 0, \qquad \text{(Model\_Stiefel)}$$

which can be equivalently reformulated [Jiang et al., 2022, Lemma 2.1] into

$$\min_{X\in \mathrm{OB}(n,k)} \|X - C\|_F^2 \quad \text{s.t. } X \geq 0, \text{ and } \|XV\|_F = 1. \qquad \text{(Model\_Oblique)}$$

Here,

- Stiefel manifold, $\mathrm{St}(n,k) \triangleq \{X \in \mathbb{R}^{n\times k} : X^\top X = I\}$.
- Oblique manifold, $\mathrm{OB}(n,k) \triangleq \{X \in \mathbb{R}^{n\times k} : \text{ all columns have unit norm}\}$.
- V is an arbitrary constant matrix satisfying $\|V\|_F = 1$ and $VV^\top > 0$ (irrelevant to X, C).

Problem II — Projection onto nonnegative Stiefel manifold

- For each Model, we conducted 20 random trials.
- Each experiment terminated successfully if solution with KKT residual $\epsilon_{kkt} = 10^{-6}$ was found.
- It failed if the maximum iteration 10,000 or maximum time 600 [s] was reached.[2]

Table: Results on Model_Stiefel

(n,k)	(60,12)			(70,14)		
	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.
RALM	**1**	**4.097**	34	**1**	**6.234**	37
REPM(LQH)	0	-	-	0	-	-
REPM(LSE)	0	-	-	0	-	-
RSQP	0.65	78.02	7	0.85	166.1	7
RIPM	**1**	**5.555**	32	**1**	**7.574**	33

[2] The success rate (Rate) over the total number of trials, the average time in seconds (Time [s]) and the average iteration number (Iter.) among the successful trials.

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Table: Results on Model_Oblique

(n,k)	(60,12)			(70,14)		
	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.
RALM	0.6	5.725	49	0.6	8.223	52
REPM(LQH)	0	-	-	0	-	-
REPM(LSE)	0	-	-	0	-	-
RSQP	0.7	44.46	5	0.5	91.38	5
RIPM	**1**	**7.134**	23	**1**	**9.268**	24

[3] The success rate (Rate) over the total number of trials, the average time in seconds (Time [s]) and the average iteration number (Iter.) among the successful trials.

Riemannian Interior Point Methods (RIPM)

Zhijian Lai, Akiko Yoshise

1. Introduction
Background
Preliminaries
2. Our proposal: Riemannian Interior Point Methods
Formulation of RIPM
Global Algorithms
Implementation
3. Numerical Experiments
4. Future work

Concluding remarks

Riemannian Constrained Optimization Problem

We consider

$$\min_{x \in \mathbb{M}} \quad f(x) \qquad \text{s.t.} \quad h(x) = 0, \text{ and } g(x) \leq 0, \tag{RCOP}$$

where $\mathbb{M}$ is a Riemannian manifold, $f : \mathbb{M} \to \mathbb{R}$, $h : \mathbb{M} \to \mathbb{R}^l$, and $g : \mathbb{M} \to \mathbb{R}^m$.

Our contributions:

1. We proposed a Riemannian version of the interior point method.
2. We proved the local superlinear/quadratic and global convergence.

Future Work

Future Work

1. **Preconditioner for linear operator equation.** Recall that we use Krylov subspace methods to solve

$$\mathcal{T}(\Delta x, \Delta y) := \begin{pmatrix} \mathcal{A}_w \Delta x + H_x \Delta y \\ H_x^* \Delta x \end{pmatrix} = \begin{pmatrix} c \\ q \end{pmatrix}, \quad (17)$$

where $\mathcal{A}_w \triangleq \operatorname{Hess}_x \mathcal{L}(w) + G_x S^{-1} Z G_x^*$. Due to the strictly complementary condition, as $k \to \infty$, the values of $S_k^{-1} Z_k$ display a huge difference of magnitude. Hence, $\Theta := G_x S^{-1} Z G_x^*$ makes $\mathcal{T}$ very ill-conditioned.

One possible way is to find another nonsingular operator $\mathcal{P}$ such that the condition number of new operator $\mathcal{P}^{-1}\mathcal{T}$ becomes smaller.

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One possible way is to find another nonsingular operator $\mathcal{P}$ such that the condition number of new operator $\mathcal{P}^{-1}\mathcal{T}$ becomes smaller.

2. **Sophisticated global strategies.** Recall that now we use
   - Merit function $\varphi(w) = \|F(w)\|^2$. (too simple)
   - Backtracking for line-search. (too simple)

The more sophisticated and robust global strategies are often based on the trust region or filter line-search method.

Reference I

[Bergmann and Herzog, 2019] Bergmann, R. and Herzog, R. (2019).
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The End

Questions? Comments?

Appendix.

Riemannian IPM (RIPM) vs. Euclidean IPM (EIPM)

1. **EIPM is a special case of RIPM** when $\mathbb{M} \equiv \mathbb{R}^n$ or $\mathbb{R}^{n\times k}$.
2. **RIPM can solve a condensed equation (18) of smaller order.**

$$\mathcal{T}(\Delta x, \Delta y) := \begin{pmatrix} \mathcal{A}_w \Delta x + H_x \Delta y \\ H_x^* \Delta x \end{pmatrix} = \begin{pmatrix} c \\ q \end{pmatrix}, \quad (18)$$

   For example, the Stiefel manifold can be used as the equality constraints; i.e., we set $h : \mathbb{M} \equiv \mathbb{R}^{n\times k} \to \mathrm{Sym}(k)$, where $h(X) = X^\top X - I_k$. Here, EIPM requires us to solve (18) of order $nk + k(k+1)/2$.
   But RIPM only requires us to solve a problem of order $nk - k(k+1)/2$, i.e., the dimension of $\mathrm{St}(n, k)$.
3. **Not all manifolds are equivalent to the smooth equality constraints.**
   For example, $\mathrm{rank}(X) = r$ is not continuous, we can not apply EIPM.

Riemannian Newton method

Riemannian Newton method: Consider

$$F(x) = 0. \tag{19}$$

Solve a linear system on $T_{x_k}M \ni v_k$:

$$\nabla F(x_k)v_k = -F(x_k),$$

then $x_{k+1} = R_{x_k}(v_k)$.

Standard Newton assumptions & Local Convergence Results:

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{(N1)There exists } x^* : F(x^*) = 0.\\ \text{(N2)}\nabla F(x^*) \text{ is nonsingular operator.}\end{array}\right\} \Rightarrow \text{superlinear[Fernandes et al., 2017]}\\ \text{(N3)}\nabla F \text{ is locally Lipschitz cont. at } x^*.\end{array}\right\} \Rightarrow \text{quadratic}[$$

Riemannian Interior Point Methods

Superlinear and Quadratic Convergence

1. Existence. There exists w^* satisfying the KKT conditions.
2. Smoothness. The functions f, g, h are smooth on $\mathcal{M}$.
3. Regularity. The set $\{\operatorname{grad} h_i(x^*) : i = 1, \cdots, l\} \cup \{\operatorname{grad} g_i(x^*) : i \in \mathcal{A}(x)\}$ is linearly independent in $T_{x^*}\mathcal{M}$.
4. Strict Complementarity. $(z^*)_i > 0$ if $g_i(x^*) = 0$ for all $i = 1, \cdots, m$.
5. Second-Order Sufficiency. $\langle \operatorname{Hess}_x \mathcal{L}(w^*)\xi, \xi \rangle > 0$ for all nonzero $\xi \in T_{x^*}\mathbb{M}$ satisfying $\langle \xi, \operatorname{grad} h_i(x^*) \rangle = 0$ for $i = 1, \cdots, l$, and $\langle \xi, \operatorname{grad} g_i(x^*) \rangle = 0$ for $i \in \mathcal{A}(x^*)$.

Proposition (L. 2022)

If assumptions (1)-(5) hold, then standard Newton assumptions (N1)-(N3) hold for KKT vector field F.

On the other hand, to keep $(s_k, z_k) \geq 0$:

- Introducing the **perturbed** complementary equation,

$$Z\Delta s + S\Delta z = -ZSe + \mu e, \tag{20}$$

  so that we are able to keep the iterates far from the boundary.

- Compute the **damped** step sizes α_k, e.g., choose $\gamma_k \in (0,1)$ and compute

$$\alpha_k := \min\left\{1, \gamma_k \min_i \left\{-\frac{(s_k)_i}{(\Delta s_k)_i} \mid (\Delta s_k)_i < 0\right\}, \gamma_k \min_i \left\{-\frac{(z_k)_i}{(\Delta z_k)_i} \mid (\Delta z_k)_i < 0\right\}\right\}, \tag{21}$$

  such that $(s_{k+1}, z_{k+1}) > 0$.

The relation of α_k and γ_k: [Yamashita and Yabe, 1996]

1. If $\gamma_k \to 1$, then $\alpha_k \to 1$.
2. If $1 - \gamma_k = O(\|F(w_k)\|)$, then $1 - \alpha_k = O(\|F(w_k)\|)$.

History of Euclidean Interior Point Method

Interior Point (IP) Method for NONLINEAR, NONCONVEX (1990-)

Early phase (1990-1995)

- Local algorithms with superlinear/ quadratic convergence [El-Bakry et al., 1996, Yamashita and Yabe, 1996].
- Global algorithms [El-Bakry et al., 1996]

Variations (1995-2010)

- Inexact Newton/ Quasi Newton IP Method
- Global strategy: *many* merit functions; linear search, or trust region, etc.

Update by Retraction

At a current point $w = (x, y, z, s)$ and direction $\Delta w = (\Delta x, \Delta y, \Delta z, \Delta s)$, the next iterate is calculated along a curve on $\mathscr{M}$, i.e.,

$$w(\alpha) := \bar{R}_w(\alpha \Delta w), \tag{22}$$

for some step length $\alpha > 0$.

By introducing

$$w(\alpha) = (x(\alpha), y(\alpha), z(\alpha), s(\alpha)), \tag{23}$$

we have

$$x(\alpha) = R_x(\alpha \Delta x),$$

and $y(\alpha) = y + \alpha \Delta y, z(\alpha) = z + \alpha \Delta z, s(\alpha) = s + \alpha \Delta s.$

Centrality conditions

Given $w_0 = (x_0, y_0, z_0, s_0)$ with $(z_0, s_0) > 0$, let
$\tau_1 := \frac{\min(Z_0 S_0 e)}{z_0^T s_0 / m}, \quad \tau_2 := \frac{z_0^T s_0}{\|F(w_0)\|}$.
Let $\gamma \in (0, 1)$ be a constant. Define centrality functions:

$$f^I(\alpha) := \min(Z(\alpha)S(\alpha)e) - \gamma\tau_1 \frac{z(\alpha)^T s(\alpha)}{m}, \tag{24}$$

$$f^{II}(\alpha) := z(\alpha)^T s(\alpha) - \gamma\tau_2 \|F(w(\alpha))\|. \tag{25}$$

For $i = I, II$, let

$$\alpha^i := \max_{\alpha \in (0,1]} \left\{\alpha : f^i(t) \geq 0, \text{ for all } t \in (0, \alpha]\right\}. \tag{26}$$

Global RIP Algorithm

1. Choose $\sigma_k \in (0, 1)$; for w_k, compute the perturbed Newton direction Δw_k with
$$\mu_k = z_k^T s_k / m \tag{27}$$
and by
$$\nabla F(w) \Delta w = -F(w) + \sigma_k \mu_k \hat{e}. \tag{28}$$
2. Step length selection.
   1. Centrality conditions: Choose $1/2 < \gamma_k < \gamma_{k-1} < 1$; compute $\alpha^i, i = I, II$, from (26); and let
   $$\bar{\alpha}_k = \min(\alpha^I, \alpha^{II}). \tag{29}$$
   2. Sufficient decreasing: Choose $\theta \in (0, 1)$, and $\beta \in (0, 1/2]$. Let $\alpha_k = \theta^t \bar{\alpha}_k$, where t is the smallest nonnegative integer such that α_k satisfies
   $$\varphi(\bar{R}_{w_k}(\alpha_k \Delta w_k)) - \varphi(w_k) \leq \alpha_k \beta \langle \operatorname{grad} \varphi_k, \Delta w_k \rangle. \tag{30}$$
3. Let $w_{k+1} = \bar{R}_{w_k}(\alpha_k \Delta w_k)$ and $k \leftarrow k + 1$.

Auxiliary Results I: Boundedness of the sequences

Given $\epsilon \geq 0$, let us define the set

$$\Omega(\epsilon) := \left\{ w \in \mathscr{M} : \epsilon \leq \varphi(w) \leq \varphi_0, \min(ZSe)/(z^T s/m) \geq \tau_1/2, z^T s/\|F(w)\| \geq \tau_2/2 \right\}.$$

Lemma (Boundedness of the sequences I, L. 2022)

If $\epsilon > 0$ and $w_k \in \Omega(\epsilon)$ for all k, then

1. *the sequence $\{z_k^T s_k\}$ and $\{(z_k)_i (s_k)_i\}$, $i = 1, 2, \ldots, m$, are all bounded above and below away from zero.*
2. *the sequence $\{z_k\}$ and $\{s_k\}$ are bounded above and component-wise bounded away from zero;*
3. *the sequence $\{w_k\}$ is bounded;*
4. *the sequence $\{\|\nabla F(w_k)^{-1}\|\}$ is bounded;*
5. *the sequence $\{\Delta w_k\}$ is bounded.*

Lemma (Boundedness of the sequences II, L. 2022)

If $\{\sigma_k\}$ is bounded away from zero. Then, $\{\bar{\alpha}_k\}$ is bounded away from zero.

Auxiliary Results II: Continuity of Some Special Scalar Fields

Lemma (L. 2022)

Let $x \in \mathcal{M}$ and A_x be a linear operator on $T_x\mathcal{M}$. Then, the values $\|\widehat{A}_x\|_2$ and $\|\hat{A}_x\|_F$ are invariant under a change of orthonormal basis; moreover,

$$\|A_x\| = \|\hat{A}_x\|_2 \leq \|\hat{A}_x\|_F. \tag{31}$$

Lemma (L. 2022)

$$x \mapsto \|\widehat{\mathrm{Hess}\, f(x)}\| \tag{32}$$

is a continuous scalar field on $\mathbb{M}$. It is true for all h_i, g_i.

$$x \mapsto \|H_x\| \text{ and } x \mapsto \|G_x\| \tag{33}$$

are continuous scalar field on $\mathbb{M}$.

Global Convergence Theorem

This theorem, now, is only proved under exponential map exp.

Lemma (Gauss [Do Carmo and Flaherty Francis, 1992, Lemma 3.5])

Let $p \in \mathcal{M}$ *and let* $v \in T_p\mathcal{M}$ *such that* $\exp_p(v)$ *is well defined. Let* $w \in T_p\mathcal{M} \approx T_v(T_p\mathcal{M})$*. Then*

$$\left\langle \mathcal{D}\exp_p(v)[v], \mathcal{D}\exp_p(v)[w] \right\rangle = \langle v, w \rangle. \tag{34}$$

Conjugate Gradients (CG) on a tangent space

Zhijian Lai, Akiko Yoshise

Input: positive definite map H on $\mathrm{T}_x\mathcal{M}$ and $b \in \mathrm{T}_x\mathcal{M}$, $b \neq 0$
Set $v_0 = 0, r_0 = b, p_0 = r_0$
For $n = 1, 2, \ldots$
  Compute Hp_{n-1} (this is the only call to H)
  $\alpha_n = \frac{\|r_{n-1}\|_x^2}{\langle p_{n-1}, Hp_{n-1}\rangle_x}$
  $v_n = v_{n-1} + \alpha_n p_{n-1}$
  $r_n = r_{n-1} - \alpha_n H p_{n-1}$
  If $r_n = 0$, **output** $s = v_n$: the solution of $Hs = b$
  $\beta_n = \frac{\|r_n\|_x^2}{\|r_{n-1}\|_x^2}$
  $p_n = r_n + \beta_n p_{n-1}$

1. Exactly the same in form of usual CG.
2. Every vectors v_n, r_n, p_n belong to tangent space $V \equiv \mathrm{T}_x\mathcal{M}$.
3. Converges very fast if H is PD with small condition number.

An Intuitive Barrier Method on Manifolds

Consider

$$\min_{x \in \mathbb{M}} \quad f(x) \quad \text{s.t.} \quad c(x) \geq 0. \qquad \text{(RCOP\_Ineq)}$$

Its logarithmic barrier function is

$$B(x;\mu) := f(x) - \mu \textstyle\sum_{i=1}^{m} \log c_i(x),$$

where $\mu > 0$. Note that the function $x \mapsto B(x;\mu)$ is differentiable on, strict $\mathcal{F} := \{x \in \mathbb{M} : c(x) > 0\}$. Its Riemannian gradient is

$$\operatorname{grad} B(x;\mu) = \operatorname{grad} f(x) - \textstyle\sum_{i=1}^{m} \frac{\mu}{c_i(x)} \operatorname{grad} c_i(x).$$

Barrier Method on Manifolds

1. Set $x_0 \in \mathbb{M}$ to a strictly feasible point, i.e., $c(x_0) > 0$, and set $\mu_0 > 0$ and $k \leftarrow 0$.
2. Check whether x_k satisfies a stopping test for (RCOP_Ineq).
3. Compute an unconstrained minimizer $x(\mu_k)$ of $B(x;\mu_k)$ with a warm starting point x_k.
4. $x_{k+1} \leftarrow x(\mu_k)$; choose $\mu_{k+1} < \mu_k$; $k \leftarrow k+1$. Return to Step 1.

Barrier Method on Manifolds

Barrier Method

Consider the following simple problem on a sphere manifold, $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$,

$$\min_{x \in \mathbb{S}^2} \quad a^T x \quad \text{s.t.} \quad x \geq 0, \tag{SP}$$

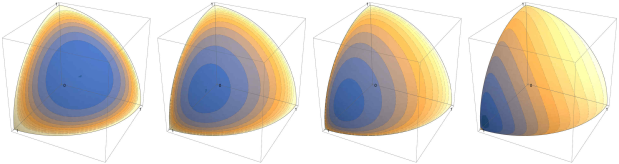
where $a = [-1, 2, 1]^T$. Its solution is $x^* = [1, 0, 0]^T$.

Now, check the KKT conditions at x (asterisks omitted below):
$\operatorname{grad} f(x) = (I_n - xx^T)a = [0, 2, 1]^T$.
The constraint $x \geq 0$ implies $c_i(x) = e_i^T x$ for $i = 1, 2, 3$;

$$\operatorname{grad} c_1(x) = (I_n - xx^T)e_1 = [0, 0, 0]^T;$$
$$\operatorname{grad} c_2(x) = (I_n - xx^T)e_2 = [0, 1, 0]^T;$$
$$\operatorname{grad} c_3(x) = (I_n - xx^T)e_3 = [0, 0, 1]^T.$$

Clearly, the multipliers $z^* = [0, 2, 1]^T$, and LICQ and strict complementarity hold.

Barrier Method on Manifolds



(a) (b) (c) (d)

Figure: Contour plots of logarithmic barrier function $B(x;\mu)$ of (SP) for (a) $\mu = 10$ (b) $\mu = 1$ (c) $\mu = 0.5$ (d) $\mu = 0.1$. The blue area indicates low values.

Barrier Method on Manifolds

Finally, we find that $\lim_{k\to\infty} x_k = x^*$ and that

$$\lim_{k\to\infty} \mu_k / c_1(x_k) = 0 = z^*_{(1)}, \lim_{k\to\infty} \mu_k / c_2(x_k) = 2 = z^*_{(2)}, \lim_{k\to\infty} \mu_k / c_3(x_k) = 1 = z$$

which are the notable features of the classical barrier method; see [Forsgren et al., 2002, Theorem 3.10 & 3.12].

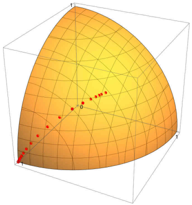


Figure: Iterates x_k of barrier method for (SP).

Barrier Method on Manifolds

Furthermore, if we denote the minimizer of $B(x;\mu)$ by either x_μ or $x(\mu)$, it must be that $\operatorname{grad} B(x_\mu;\mu) = 0$.

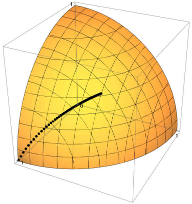


Figure: Existence of a central path for (SP).